

Equivalence of linear stabilities of elliptic triangle solutions of the planar charged and classical three-body problems

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Abstract

In this paper, we prove that the linearized system of elliptic triangle homographic solution of planar charged three-body problem can be transformed to that of the elliptic equilateral triangle solution of the planar classical three-body problem. Consequently, the results of Martínez, Samà and Simó ([15] in J. Diff. Equa.) of 2006 and results of Hu, Long and Sun ([6] in arXiv.org) of 2012 can be applied to these solutions of the charged three-body problem to get their linear stability.

Keywords: charged three-body problem, linear stability, equivalence, elliptic relative equilibria.

AMS Subject Classification: 70F07, 70H14, 37J45

1 Main results

We consider the charged planar three-body problem concerns of 3 point particles endowed with a positive mass $m_j \in \mathbf{R}^+ = \{r \in \mathbf{R} \mid r > 0\}$ and an electrostatic charge of any sign $e_j \in \mathbf{R}, j = 1, 2, 3$, moving under the influence of the respective Newtonian and Coulombian force. Denote by $q_1, q_2, q_3 \in \mathbf{R}^2$ the position vectors of the three particles respectively. By Newton's second law, the law of universal gravitation and Coulombian's law, the system of equations for this problem is

$$m_i \ddot{q}_i = \sum_{j \neq i} \frac{m_i m_j - e_i e_j}{|q_i - q_j|^3} (q_j - q_i) = \frac{\partial U(q)}{\partial q_i}, \quad \text{for } i = 1, 2, 3, \quad (1.1)$$

where $U(q) = U(q_1, q_2, q_3) = \sum_{1 \leq i < j \leq 3} \frac{m_i m_j - e_i e_j}{|q_i - q_j|}$ is the potential or force function by using the standard norm $|\cdot|$ of vectors in \mathbf{R}^2 . For periodic solutions with period 2π , the system (1.1) is the Euler-Lagrange equation of the action functional

$$\mathcal{A}(q) = \int_0^{2\pi} \left[\sum_{i=1}^3 \frac{m_i |\dot{q}_i(t)|^2}{2} + U(q(t)) \right] dt$$

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defined on the loop space $W^{1,2}(\mathbf{R}/(2\pi\mathbf{Z}), \hat{\mathcal{X}})$, where

$$\hat{\mathcal{X}} = \left\{ q = (q_1, q_2, q_3) \in (\mathbf{R}^2)^3 \mid \sum_{i=1}^3 m_i q_i = 0, q_i \neq q_j, \forall i \neq j \right\}$$

is the configuration space of the planar three-body problem. Periodic solutions of (1.1) correspond to critical points of the action functional \mathcal{A} .

It is a well-known fact that (1.1) can be reformulated as a Hamiltonian system. Let $p_1, p_2, p_3 \in \mathbf{R}^2$ be the momentum vectors of the particles respectively. The Hamiltonian system corresponding to (1.1) is

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \text{for } i = 1, 2, 3, \quad (1.2)$$

with Hamiltonian function

$$H(p, q) = H(p_1, p_2, p_3, q_1, q_2, q_3) = \sum_{i=1}^3 \frac{|p_i|^2}{2m_i} - U(q_1, q_2, q_3). \quad (1.3)$$

Note that if all charges are zero, the problem reduces to the classical Newtonian one. The charged problem has a more complicated dynamical behavior.

Central configurations are basic topics which help understanding the complexity of the charged problem. It is well known that, in the classical Newtonian three-body problem, there are five central configurations: two of them are equilateral triangles and three of them are collinear. In the charged problem, Pérez-Chavela, Sarri, Susin and Yan ([17], 1996) proved that there might exist at most five collinear central configurations under some constraints of the parameters (masses and quantities of electric charge). They also proved that, if there exist non-collinear central configurations, the shape of such central configuration is determined by the masses and quantities of electric charge, and hence may not be an equilateral triangle in general.

In the charged three-body problem, when the three bodies form a central configuration and each of which move along a Keplerian orbit with eccentricity $e \in [0, 1)$, we call such solutions of the system (1.1) *elliptic relative equilibria*. Specially when $e = 0$, the Keplerian elliptic motion becomes circular motion, which are called *relative equilibria* traditionally.

Our main concern in this paper is the linear stability of these homographic solutions. For the planar three-body problem with masses $m_1, m_2, m_3 > 0$ and quantities of charges $e_1, e_2, e_3 \in \mathbf{R}$, it turns out that the stability of these elliptic triangular solutions depends on two parameters, namely the eccentricity $e \in [0, 1)$ and the mass parameter $\beta \in [0, 9]$ defined by

$$\beta = \frac{36(m_1 m_2 \sin^2 \theta_3 + m_1 m_3 \sin^2 \theta_2 + m_2 m_3 \sin^2 \theta_1)}{(m_1 + m_2 + m_3)^2}, \quad (1.4)$$

where $\theta_i, i = 1, 2, 3$ are the three inner angles of the central configuration formed by the three particles. When the central configuration is an equilateral triangle, i.e., $\theta_i = \frac{\pi}{3}$ for all $i = 1, 2, 3$, then β in (1.4) here coincides with β in (1.4) of [6] in the Newtonian case.

In [17] of 1996 of Pérez-Chavela, Saari, Susin, and Yan, and [1] of 2008 of Alfaro and Pérez-Chavela, the authors considered the relative equilibria and their stabilities of three charged bodies moving under the influence of the respective Newtonian and Colombian forces. In Section 4 of [17], the authors proved that, in the charged three-body problem, if $\delta_{ij} > 0$ for $1 \leq i < j \leq 3$ (defined by (2.3) below), and $\delta_{12}^{1/3}$, $\delta_{23}^{1/3}$, and $\delta_{31}^{1/3}$ are the lengths of three sides of some triangle, there exists two non-collinear relative equilibria (one is mirror symmetric to the other). Moreover, in Theorem 2 of [1] (cf. p. 1940), the authors proved that, a non-collinear relative equilibrium of

charged three-body problem is both linearly stable and non-degenerate if and only if the masses and charges satisfy the condition $\beta < 1$. This is a special case with the eccentricity $e = 0$ of the stability problem of elliptic relative equilibria of charged three bodies.

The linear stability of relative equilibria in the Newtonian case were known more than a century ago and it is due to Gascheau ([4], 1843) and Routh ([19], 1875) independently. Further studies can be found in works of Danby ([3], 1964) and Roberts ([18], 2002). In 2005, Meyer and Schmidt (cf. [16]) used heavily the central configuration nature of the elliptic Lagrangian orbits and decomposed the fundamental solution of the elliptic Lagrangian solution into two parts symplectically, one of which is the same as that of the Keplerian solution and the other is the essential part for the stability.

Here we point out first that a flat homographic solution must be planar in the charged case. The proof in [12] (cf. pp. 39-41) and [11] (cf. Theorem 2.6) for the Newtonian case works also for our problem with some minor modifications when $\delta_{ij} \neq 0$ for $1 \leq i < j \leq 3$. This was already known by Proposition 1 of [1] in the charged n -body problem when every δ_{ij} is positive.

In this paper, following the central configuration coordinate method of Meyer and Schmidt in [16], we linearize the Hamiltonian system (1.2)-(1.3) of the charged three bodies near an elliptic relative equilibrium. We found that the essential part of this linearized Hamiltonian system coincides with that of the linearized system of the corresponding Hamiltonian system for the Newtonian case at the elliptic Lagrangian equilateral triangle solution (cf. [9]), i.e., the system (17) on p.275 of [16] (cf. also (2.19) in [6]) depending on the eccentricity $e \in [0, 1)$ and $\beta \in [0, 9]$ given by (1.4) with $\theta_i = \pi/3$ for $i = 1, 2$, and 3 . Moreover, as proved in the Appendix below the full range of the parameter β of (1.4) is $[0, 9]$ for all admissible quantities of parameters which forms a non-collinear elliptic relative equilibria. Our main result in this paper is the following

Theorem 1.1 *Let $q(t) = (r(t)R(\theta(t))a_1, r(t)R(\theta(t))a_2, r(t)R(\theta(t))a_3)$ be an elliptic relative equilibrium of the system (1.1) with $\delta_{ij} > 0$ for $1 \leq i < j \leq 3$, where (a_1, a_2, a_3) is a non-collinear central configuration of the same charged three-body problem. Then the linearized system of (1.1) at q can be transformed to the linearized system of the classical three-body problem at the elliptic equilateral triangle solution with the same eccentricity $e \in [0, 1)$ and $\beta \in [0, 9]$ given by (1.4).*

In 2004-2006, Martínez, Samà and Simó ([13],[14],[15], 2004-2006) studied the stability of the elliptic Lagrangian solution of the classical three body problem when $e > 0$ is small enough by using normal form theory, and $e < 1$ and close to 1 enough by using blow-up technique in general homogeneous potential. They further gave a rather complete bifurcation diagram of the problem numerically and a beautiful figure (Fig. 5 in [15]) was drawn there for the full (β, e) range, which we repeat here as Figure 1. Here the regions I, II, III, IV, V and VI are EE, EE, EH, HH, HH and CS respectively.

Denote the fundamental solution of the linearized Hamiltonian system of the essential part of the elliptic relative equilibrium by $\gamma_{\beta,e}(t)$ for $t \in [0, 2\pi]$. Let \mathbf{U} denote the unit circle in the complex plane \mathbf{C} . As in [15], the following notations for the different parameter regions are used in Figure 1:

- EE (elliptic-elliptic), if $\gamma_{\beta,e}(2\pi)$ possesses two pairs of eigenvalues in $\mathbf{U} \setminus \mathbf{R}$;
- EH (elliptic-hyperbolic), if $\gamma_{\beta,e}(2\pi)$ possesses a pair of eigenvalues in $\mathbf{U} \setminus \mathbf{R}$ and a pair of eigenvalues in $\mathbf{R} \setminus \{0, \pm 1\}$;
- HH (hyperbolic-hyperbolic), if $\sigma(\gamma_{\beta,e}(2\pi)) \subset \mathbf{R} \setminus \{0, \pm 1\}$;
- CS (complex-saddle), if $\sigma(\gamma_{\beta,e}(2\pi)) \subset \mathbf{C} \setminus (\mathbf{U} \cup \mathbf{R})$.

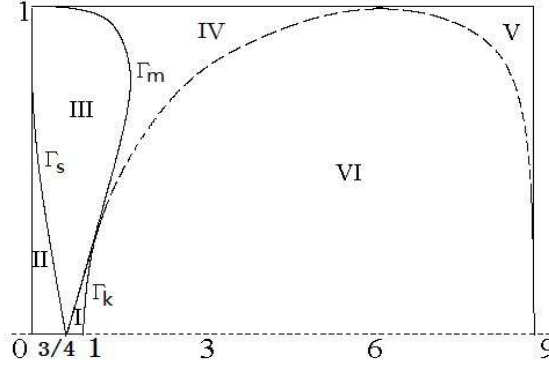


Figure 1: Stability bifurcation diagram of elliptic relative equilibria of the charged and classical three-body problem in the (β, e) rectangle $[0, 9] \times [0, 1)$.

In [7] and [8] of 2009-2010, Hu and Sun found a new way to relate the stability problem to the iterated Morse indices. Recently, by observing new phenomenons and discovering new properties of elliptic Lagrangian solution, in the joint paper [6] of Hu, Long and Sun, the linear stability of elliptic Lagrangian solution is completely solved analytically by index theory (cf. [10]) and the new results are related directly to (β, e) in the full parameter rectangle $\Theta = [0, 9] \times [0, 1)$.

Then by our Theorem 1.1 and Theorem 2.3 below, which yields the minimization property of these elliptic relative equilibria and then the values of the corresponding Morse indices of the corresponding functional at $\beta = 0$. Thus results in [6] as well as [15] can be applied to the linear stability problem of the elliptic relative equilibria of the charged 3-body problem. Specially following [6] we have

Corollary 1.2 (i) *The elliptic relative equilibrium $q_{\beta,e}$ of the charged 3-body problem for every $(\beta, e) \in [0, 9] \times [0, 1)$ possesses Morse index $i_1(q) = 0$. The essential part $\gamma_{\beta,e}$ of the fundamental solution of the linearized system of (1.1) at $q_{\beta,e}$ is non-degenerate, i.e., $\nu_1(\gamma_{\beta,e}) = 0$, if $\beta > 0$, and possesses nullity $\nu_1(\gamma_{\beta,e}) = 3$ when $\beta = 0$.*

(ii) *In the (β, e) rectangle $\Theta = (0, 9] \times [0, 1)$, there exist three distinct continuous curves from left to right (cf. Figure 1): Γ_s and Γ_m going up from $(3/4, 0)$ with tangents $-\sqrt{33}/4$ and $\sqrt{33}/4$ respectively and converges to the point $(0, 1)$, and Γ_k going up from $(1, 0)$ and converges to the point $(0, 1)$ as e increase to 1; each of them intersect every horizontal segment $e = \text{constant} \in [0, 1)$ only once. The linear stability pattern of the elliptic solution changes when (β, e) passes through one of these three curves Γ_s , Γ_m and Γ_k . More precisely, these three curves separate Θ into sub-regions of linear stability: EE on the left hand side of Γ_s , EH in between Γ_s and Γ_m , EE in between Γ_m and Γ_k , and hyperbolic on the right hand side of Γ_k .*

(iii) *When $e = 0$, the relative equilibrium $q_{\beta,e}$ is linearly stable if $0 < \beta < 1$, spectrally stable and linearly unstable when $\beta = 1$, and hyperbolic when $\beta > 1$.*

Proof. By our Theorems 1.1 and 2.3, the index properties of $q_{\beta,e}$ at $\beta = 0$ are established, i.e., (i) holds. Therefore results in [6] can be applied to get the corollary. Then (ii) and (iii) follow from Theorems 1.2 and 1.5-1.8 of [6]. ■

Note first that more stability information for (β, e) located precisely on these three curves can be found in Theorem 1.8 of [6], and is omitted here. Note also that when $e = 0$, by (i) and (iii) of our Corollary 1.2 the relative equilibrium $q_{\beta,e}$ is non-degenerate whenever $\beta > 0$, and is linearly stable if and only if $0 < \beta < 1$. Therefore our Corollary 1.2 generalizes specially Theorem 2 on p.1940 of [1].

This paper is arranged as follows. In Section 2, we study elliptic relative equilibria of the charged three body problem and their relations with the corresponding central configurations, and their variational minimization property. Then in Section 3 we give the proof of Theorem 1.1.

2 Central configurations and minimizing property of the relative equilibria of the charged problem

We need the concept of central configurations in the charged problem as in [17] similar to the Newtonian case.

Definition 2.1 *A configuration $a = (a_1, a_2, \dots, a_n) \in (\mathbf{R}^k)^n$ with $a_i \neq a_j, \forall 1 \leq i < j \leq n$ is a central configuration for the given mass $m = (m_1, m_2, \dots, m_n) \in (\mathbf{R}^+)^n$ and the quantities of charges $e = (e_1, \dots, e_n) \in \mathbf{R}^n$, if there exists some $\lambda \in \mathbf{R}$ such that (q, λ) is a solution of the algebraic system*

$$\lambda M a + \frac{\partial U(a)}{\partial q} = 0, \quad (2.1)$$

with $M = \text{diag}(m_1 I_k, \dots, m_n I_k)$. By the homogeneity of U of degree -1 , (2.1) implies

$$\lambda = U(a)/(M a \cdot a). \quad (2.2)$$

In this paper, we only need the definition with $k = 2$. Let's define

$$\delta_{ij} = \frac{m_i m_j - e_i e_j}{m_i m_j} = 1 - \frac{e_i e_j}{m_i m_j}. \quad (2.3)$$

Proposition 2.2 *Let $(a_1, a_2, \dots, a_n) \in (\mathbf{R}^2)^n \setminus \{0\}$ be a configuration for mass $m = (m_1, m_2, \dots, m_n) \in (\mathbf{R}^+)^n$ and the quantities of charges $e = (e_1, \dots, e_n) \in \mathbf{R}^n$. Without loss of generality, we set*

$$I(a) = \sum_{i=1}^n m_i |a_i|^2 = 1. \quad (2.4)$$

Let $(Z(t), z(t))^T \in (\mathbf{R}^2)^2$ be a solution of the Kepler central force problem with Hamiltonian function

$$H_K = \frac{|Z|^2}{2} - \frac{\mu}{z}, \quad z, Z \in \mathbf{R}^2, \quad (2.5)$$

where

$$\mu = \sum_{1 \leq i < j \leq n} \frac{m_i m_j - e_i e_j}{|a_i - a_j|} = \sum_{1 \leq i < j \leq n} \frac{m_i m_j \delta_{ij}}{|a_i - a_j|}. \quad (2.6)$$

Write $z(t) = r(t)(\cos \theta(t), \sin \theta(t))^T$ for all t . For all $1 \leq i \leq n$ define

$$q_i(t) = r(t)R(\theta(t))a_i, \quad p_i(t) = m_i \dot{q}_i(t) = m_i[\dot{r}(t)R(\theta(t)) + r(t)\dot{\theta}(t)JR(\theta(t))]a_i, \quad (2.7)$$

where $R(\theta)$ is the rotation matrix with angular θ . Then $(p, q) = (p_1(t), \dots, p_n(t), q_1(t), \dots, q_n(t))$ form a solution of the charged n -body problem if and only if (a_1, a_2, \dots, a_n) is a central configuration of the charged n -body problem for mass $m = (m_1, m_2, \dots, m_n)$ and the quantities of charges $e = (e_1, \dots, e_n)$.

Proof. It suffices to prove that the configuration a satisfies (2.1) with some constant λ given by (2.2) if and only if (p, q) given by (2.7) satisfies the first system on \dot{p}_i s in (1.2), i.e.,

$$\dot{p}_i = U_{q_i}(q), \quad (2.8)$$

with

$$U(q) = \sum_{1 \leq i < j \leq n} \frac{m_i m_j \delta_{ij}}{|q_i - q_j|}. \quad (2.9)$$

Here the second system on \dot{q}_i s in (1.2) is automatically satisfied by (2.7).

In fact, firstly, by the definition of q_i s in (2.7) we have

$$\begin{aligned} U_{q_i}(q) &= - \sum_{j=1, j \neq i}^n \frac{m_i m_j \delta_{ij}}{|q_i - q_j|^3} (q_i - q_j) \\ &= - \sum_{j=1, j \neq i}^n \frac{m_i m_j \delta_{ij}}{r(t)^3 |a_i - a_j|^3} r(t) R(\theta(t)) (a_i - a_j) \\ &= \frac{1}{r(t)^2} R(\theta(t)) \left[- \sum_{j=1, j \neq i}^n \frac{m_i m_j \delta_{ij}}{|a_i - a_j|^3} (a_i - a_j) \right] \\ &= \frac{1}{r(t)^2} R(\theta(t)) U_{q_i}(a). \end{aligned} \quad (2.10)$$

On the other hand, by the definition of p_i s in (2.7) we have

$$\begin{aligned} \dot{p}_i &= m_i [\ddot{r}(t) R(\theta(t)) + 2\dot{r}(t)\dot{\theta}(t) J R(\theta(t)) + r(t)\ddot{\theta}(t) J R(\theta(t)) + r(t)\dot{\theta}(t)^2 J^2 R(\theta(t))] a_i \\ &= m_i [\ddot{r}(t) + (2\dot{r}(t)\dot{\theta}(t) + r(t)\ddot{\theta}(t)) J - r(t)\dot{\theta}(t)^2] R(\theta(t)) a_i. \end{aligned} \quad (2.11)$$

We know that the Kepler orbit $z(t)$ satisfies

$$\ddot{z} = -\frac{\mu}{r(t)^3} z(t) \quad (2.12)$$

with $r(t) = |z(t)|$. By a computation similar to that in Section 1.2 in [11], we have

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}, \quad r^2\dot{\theta} = c. \quad (2.13)$$

Differentiating the second identity in (2.13), we obtain

$$r(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0. \quad (2.14)$$

Then by (2.11)-(2.14), and the fact $r(t) \neq 0$, we have

$$\dot{p}_i = m_i [\ddot{r}(t) - r(t)\dot{\theta}(t)^2] R(\theta(t)) a_i = \frac{1}{r(t)^2} R(\theta(t)) (-\mu) m_i a_i. \quad (2.15)$$

Thus by (2.10) and (2.15), (p, q) satisfies (2.8) if and only if a satisfies (2.1) with $\lambda = \mu$. ■

In [17] and [1], if $\delta_{12}, \delta_{23}, \delta_{31} > 0$ and satisfy the constraint

$$\delta_{ij}^{1/3} + \delta_{jk}^{1/3} > \delta_{ki}^{1/3}, \quad (2.16)$$

where (i, j, k) permutes cyclically in $(1, 2, 3)$, there exist an elliptic triangle solution of equation (1.1) with the following form:

$$q(t) = (r(t)R(\theta(t))a_1, r(t)R(\theta(t))a_2, r(t)R(\theta(t))a_3), \quad (2.17)$$

where (a_1, a_2, a_3) forms a central configuration of the charged three-body problem. Moreover, a triangle (non-collinear) central configuration of three charged bodies must satisfy

$$|q_1 - q_2| : |q_2 - q_3| : |q_3 - q_1| = \sqrt[3]{\delta_{12}} : \sqrt[3]{\delta_{23}} : \sqrt[3]{\delta_{31}}. \quad (2.18)$$

In the following, we will always suppose $\delta_{12}, \delta_{23}, \delta_{31} > 0$ and (2.16) holds.

A different important way to access the n -body problem is to study its corresponding variational functional. For a closed curve $u : S^1 \rightarrow \mathbf{R}^2 \setminus \{0\}$, we denote its index around the origin by $\text{ind}(u, 0) = \text{deg}(u, 0)$. Let $P = \{(1, 2), (2, 3), (3, 1)\}$. For $k = (k_{12}, k_{23}, k_{31}) \in \mathbf{Z}^3$ and $\tau > 0$, define

$$\Omega_{\tau, k} = \{q = (q_1, q_2, q_3) \in C^\infty(\mathbf{R}/(\tau\mathbf{Z}), \hat{\mathcal{X}}) \mid \text{ind}(q_i - q_j, 0) = k_{ij}, \forall (i, j) \in P\}. \quad (2.19)$$

Then we let $X_{\tau, k}$ be the $W^{1,2}$ completion of $\Omega_{\tau, k}$. We define a functional on $X_{\tau, k}$:

$$f(q) = \int_0^\tau \left\{ \frac{1}{2} T_q(t) + U(q(t)) \right\} dt, \quad \forall q \in X_{\tau, k}, \quad (2.20)$$

where $T_q(t) = \sum_{i=1}^3 m_i |\dot{q}_i(t)|^2$ and $U(q) = \sum_{i < j} m_i m_j \delta_{ij} / |q_i - q_j|$. Then following [5], [20] and [21] we have the theorem below.

Theorem 2.3 *Let $m = (m_1, m_2, m_3) \in (\mathbf{R}^+)^3$, $\tau > 0$ and $k = (1, 1, 1)$ or $k = (-1, -1, -1)$. Suppose $\delta_{12}, \delta_{23}, \delta_{31} > 0$ and (2.16) holds. Then the following holds:*

(i) *The minimum of f on $X_{\tau, k}$ is given by*

$$\inf_{q \in X_{\tau, k}} f(q) = \left(\sum_{(i, j) \in P} m_i m_j \delta_{ij}^{2/3} \right) 3(2^{-1/3}) \pi^{2/3} \tau^{1/3}. \quad (2.21)$$

(ii) *The elliptic triangle solutions of the charged three-body problem attains the minimum of f on $X_{\tau, k}$.*

(iii) *Every regular, i.e., C^2 smooth, minimizer of f on $X_{\tau, k}$ must be an elliptic triangle solution.*

Here recall that the elliptic triangle solution is given by $q(t) = r(t)R(\theta(t))a$ as in (2.7) for $n = 3$ such that $a = (a_1, a_2, a_3)$ forms a nonlinear central configuration. Moreover, without lose of generality, we normalize the three masses by

$$m_1 + m_2 + m_3 = 1. \quad (2.22)$$

Proof. Note first that $q \in W^{1,2}$ implies that q is C^0 and \dot{q} exists a.e. in t . Thus from $\sum_{i=1}^3 m_i q_i(t) = 0$, we obtain $\sum_{i=1}^3 m_i \dot{q}_i(t) = 0$ a.e. in t . On such t applying Largrange's identity (cf. p.73 of [2]) to $\dot{q}(t)$, we obtain

$$\sum_{i=1}^3 m_i |\dot{q}_i(t)|^2 = \sum_{(i, j) \in P} m_i m_j |\dot{q}_i(t) - \dot{q}_j(t)|^2.$$

This yields

$$f(q) = \sum_{(i, j) \in P} m_i m_j \int_0^\tau \left(\frac{|\dot{q}_i - \dot{q}_j|^2}{2} + \frac{\delta_{ij}}{|q_i - q_j|} \right) dt. \quad (2.23)$$

We define

$$\tilde{q}_{ij} = \frac{q_i - q_j}{\delta_{ij}^{1/3}}, \quad \forall 1 \leq i \neq j \leq 3, \quad (2.24)$$

and then we have

$$f(q) = \sum_{(i,j) \in P} m_i m_j \delta_{ij}^{2/3} \int_0^\tau \left(\frac{|\dot{\tilde{q}}_{ij}|^2}{2} + \frac{1}{|\tilde{q}_{ij}|} \right) dt. \quad (2.25)$$

For each $(i, j) \in P$, by both Theorem 1.1 and formula (2.2) of W.Gordon [5], we obtain

$$\mathcal{P}(q) = \int_0^\tau \left(\frac{|\dot{\tilde{q}}_{ij}|^2}{2} + \frac{1}{|\tilde{q}_{ij}|} \right) dt \geq 3(2^{-1/3})\pi^{2/3}\tau^{1/3}. \quad (2.26)$$

Thus we have

$$f(q) \geq \left(\sum_{(i,j) \in P} m_i m_j \delta_{ij}^{2/3} \right) 3(2^{-1/3})\pi^{2/3}\tau^{1/3} \quad (2.27)$$

for all $q \in X_{\tau,k}$.

On the other hand, for every elliptic triangle solution

$$q = (q_1, q_2, q_3) = (r(t)R(\theta(t))a_1, r(t)R(\theta(t))a_2, r(t)R(\theta(t))a_3)$$

and all $t \in \mathbf{R}$, by (2.18), we have

$$|q_1 - q_2| : |q_2 - q_3| : |q_3 - q_1| = |a_1 - a_2| : |a_2 - a_3| : |a_3 - a_1| = \sqrt[3]{\delta_{12}} : \sqrt[3]{\delta_{23}} : \sqrt[3]{\delta_{31}}.$$

Using the definition of \tilde{q}_{ij} of (2.24), we have

$$\begin{aligned} |\tilde{q}_{12}(t)| &= |\tilde{q}_{23}(t)| = |\tilde{q}_{31}(t)| \equiv \rho(t), \\ \delta_{12}^{1/3} \tilde{q}_{12}(t) + \delta_{23}^{1/3} \tilde{q}_{23}(t) + \delta_{31}^{1/3} \tilde{q}_{31}(t) &= 0. \end{aligned}$$

Therefore from the system (1.1) for $\bar{m} = m_1 + m_2 + m_3 = 1$, we obtain

$$\begin{aligned} \ddot{q}_i &= \sum_{1 \leq j \leq 3, j \neq i} \frac{m_j \delta_{ij}}{[\delta_{ij}^{1/3} \rho(t)]^3} (q_j - q_i) \\ &= \sum_{1 \leq j \leq 3, j \neq i} \frac{m_j q_j(t)}{\rho(t)^3} - (1 - m_i) \frac{q_i(t)}{\rho(t)^3} \\ &= \frac{\sum_{1 \leq i \leq 3} m_i q_i(t)}{\rho(t)^3} - \frac{q_i(t)}{\rho(t)^3} \\ &= -\frac{q_i(t)}{\rho(t)^3}, \end{aligned}$$

where we have used (2.22) and the fact $\sum_{1 \leq i \leq 3} m_i q_i(t) = 0$. Therefore, we have

$$\ddot{\tilde{q}}_{ij}(t) = \frac{\dot{q}_i - \dot{q}_j}{\delta_{ij}^{1/3}} = -\frac{\tilde{q}_{ij}(t)}{\rho(t)^3} = -\frac{\tilde{q}_{ij}(t)}{|\tilde{q}_{ij}(t)|^3}, \quad (2.28)$$

for all $t \in \mathbf{R}$. Then by Theorem 1.1 of [5], the action \mathcal{P} of (2.26) attains its minimum at the Kepler solution \tilde{q}_{ij} . By (2.25), this proves that the functional f attains its minimum at the elliptic triangle solutions. Thus we have proved (i) and (ii).

Now (iii) follows from (i) and the proof in [5] immediately. \blacksquare

3 The essential part of the fundamental solution of the elliptic orbit of the charged problem

Proposition 6 of [1] states that we can have triangular relative equilibria, where the triangle has any shape. Then we can fix a triangle as a central configuration of the charged three-body problem for some masses $m \in (\mathbf{R}^+)^3$ and quantities of charges $e \in \mathbf{R}^3$. Denote by $\theta_1, \theta_2, \theta_3$ the three inner angles respectively, see Figure 2. We have the following theorem.

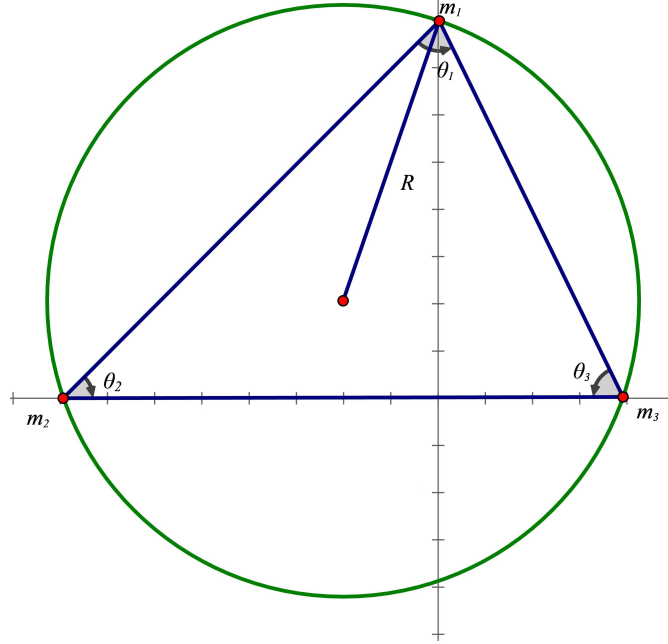


Figure 2: The nonlinear central configuration of three charged bodies

Theorem 3.1 *The linearized system of (1.2) with Hamiltonian function (1.3) near the elliptic triangle solution $q(t)$ of (2.17) can be transformed to*

$$\begin{pmatrix} \dot{\bar{Z}} \\ \dot{\bar{z}} \\ \dot{\bar{W}} \\ \dot{\bar{w}} \end{pmatrix} = \begin{pmatrix} J\bar{B}_1(\theta) & O \\ O & J\bar{B}_2(\theta) \end{pmatrix} \begin{pmatrix} \bar{Z} \\ \bar{z} \\ \bar{W} \\ \bar{w} \end{pmatrix}, \quad (3.1)$$

where e is the eccentricity, and we define

$$\beta = 36\alpha^2 = 36(m_2m_3 \sin^2 \theta_1 + m_3m_1 \sin^2 \theta_2 + m_1m_2 \sin^2 \theta_3). \quad (3.2)$$

And

$$\begin{pmatrix} \dot{\bar{W}} \\ \dot{\bar{w}} \end{pmatrix} = J\bar{B}_2(\theta) \begin{pmatrix} \bar{W} \\ \bar{w} \end{pmatrix} \quad (3.3)$$

is the essential part of the linearized system of (3.1) with

$$\bar{B}_2(\theta) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & \frac{2e \cos \theta - 1 - \sqrt{9-\beta}}{2(1+e \cos \theta)} & 0 \\ 1 & 0 & 0 & \frac{2e \cos \theta - 1 + \sqrt{9-\beta}}{2(1+e \cos \theta)} \end{pmatrix}, \quad (3.4)$$

The rest of this paper focuses on the proof of this theorem.

In [16] (cf. p.275), Meyer and Schmidt give the essential part of the fundamental solution of the elliptic Lagrangian orbit. This method also can be found in [11]. For elliptic solutions of the charged problem, we will follow their method.

Suppose the coordinates of the three particles are given by

$$\hat{a}_1 = (0, y)^T, \quad \hat{a}_2 = (-x_1, 0)^T, \quad \hat{a}_3 = (x_2, 0)^T, \quad (3.5)$$

where $x_1, x_2, y > 0$. Recall that $\theta_1, \theta_2, \theta_3$ are the three inner angles respectively. For convenience, we denote by R the radius of the circumscribed circle of the triangle. Then we have

$$|\hat{a}_1 - \hat{a}_2| = 2R \sin \theta_3, \quad |\hat{a}_2 - \hat{a}_3| = 2R \sin \theta_1, \quad |\hat{a}_3 - \hat{a}_1| = 2R \sin \theta_2, \quad (3.6)$$

and

$$\hat{a}_1 = (0, 2R \sin \theta_2 \sin \theta_3)^T, \quad \hat{a}_2 = (-2R \cos \theta_2 \sin \theta_3, 0)^T, \quad \hat{a}_3 = (2R \sin \theta_2 \cos \theta_3, 0)^T. \quad (3.7)$$

By (2.18) and (3.6), we have

$$\sin \theta_1 : \sin \theta_2 : \sin \theta_3 = \sqrt[3]{\delta_{23}} : \sqrt[3]{\delta_{31}} : \sqrt[3]{\delta_{12}}. \quad (3.8)$$

Then by (2.22), the center of mass of the three particles is

$$c(m) = m_1 \hat{a}_1 + m_2 \hat{a}_2 + m_3 \hat{a}_3 = \begin{pmatrix} 2R(-m_2 \cos \theta_2 \sin \theta_3 + m_3 \sin \theta_2 \cos \theta_3) \\ 2Rm_1 \sin \theta_2 \sin \theta_3 \end{pmatrix}. \quad (3.9)$$

Let

$$a_i = \frac{\hat{a}_i - c(m)}{2R\alpha} \quad \forall i = 1, 2, 3, \quad (3.10)$$

and some $\alpha > 0$. Then the center of masses of a_i s is at the origin, and we have

$$a_1 = \frac{1}{\alpha} \begin{pmatrix} m_2 \cos \theta_2 \sin \theta_3 - m_3 \sin \theta_2 \cos \theta_3 \\ (m_2 + m_3) \sin \theta_2 \sin \theta_3 \end{pmatrix}, \quad (3.11)$$

$$\begin{aligned} a_2 &= \frac{1}{\alpha} \begin{pmatrix} -(m_1 + m_3) \cos \theta_2 \sin \theta_3 - m_3 \sin \theta_2 \cos \theta_3 \\ -m_1 \sin \theta_2 \sin \theta_3 \end{pmatrix}, \\ &= \frac{1}{\alpha} \begin{pmatrix} -m_1 \cos \theta_2 \sin \theta_3 - m_3 \sin(\theta_2 + \theta_3) \\ -m_1 \sin \theta_2 \sin \theta_3 \end{pmatrix}, \end{aligned} \quad (3.12)$$

$$\begin{aligned} a_3 &= \frac{1}{\alpha} \begin{pmatrix} m_2 \cos \theta_2 \sin \theta_3 + (m_1 + m_2) \sin \theta_2 \cos \theta_3 \\ -m_1 \sin \theta_2 \sin \theta_3 \end{pmatrix} \\ &= \frac{1}{\alpha} \begin{pmatrix} m_2 \sin(\theta_2 + \theta_3) + m_1 \sin \theta_2 \cos \theta_3 \\ -m_1 \sin \theta_2 \sin \theta_3 \end{pmatrix}. \end{aligned} \quad (3.13)$$

To determine α , we set the momentum of a_i s to be 1, by (2.22) and

$$\theta_1 + \theta_2 + \theta_3 = \pi, \quad (3.14)$$

which yield

$$\begin{aligned} 1 &= m_1 |a_1|^2 + m_2 |a_2|^2 + m_3 |a_3|^2 \\ &= \frac{1}{\alpha^2} [m_1(m_2^2 \sin^2 \theta_3 + m_3^2 \sin^2 \theta_2 - 2m_2 m_3 \sin \theta_2 \sin \theta_3 \cos(\theta_2 + \theta_3)) \end{aligned}$$

$$\begin{aligned}
& +m_2(m_1^2 \sin^2 \theta_3 + m_3^2 \sin^2(\theta_2 + \theta_3) + 2m_3m_1 \sin(\theta_2 + \theta_3) \cos \theta_2 \sin \theta_3) \\
& +m_3(m_1^2 \sin^2 \theta_2 + m_2^2 \sin^2(\theta_2 + \theta_3) + 2m_1m_2 \sin(\theta_2 + \theta_3) \sin \theta_2 \cos \theta_3)] \\
= & \frac{1}{\alpha^2} [m_2m_3(m_2 + m_3) \sin^2 \theta_1 + m_3m_1(m_3 + m_1) \sin^2 \theta_2 + m_1m_2(m_1 + m_2) \sin^2 \theta_3 \\
& + 2m_1m_2m_3(\cos \theta_1 \sin \theta_2 \sin \theta_3 + \sin \theta_1 \cos \theta_2 \sin \theta_3 + \sin \theta_1 \sin \theta_2 \cos \theta_3)] \\
= & \frac{1}{\alpha^2} [m_2m_3(1 - m_1) \sin^2 \theta_1 + m_3m_1(1 - m_2) \sin^2 \theta_2 + m_1m_2(1 - m_3) \sin^2 \theta_3 \\
& + 2m_1m_2m_3(\cos \theta_1 \sin \theta_2 \sin \theta_3 + \sin \theta_1 \cos \theta_2 \sin \theta_3 + \sin \theta_1 \sin \theta_2 \cos \theta_3)] \\
= & \frac{1}{\alpha^2} [m_2m_3 \sin^2 \theta_1 + m_3m_1 \sin^2 \theta_2 + m_1m_2 \sin^2 \theta_3] \\
& + \frac{m_1m_2m_3}{\alpha^2} [2 \cos \theta_1 \sin \theta_2 \sin \theta_3 + 2 \sin \theta_1 \cos \theta_2 \sin \theta_3 + 2 \sin \theta_1 \sin \theta_2 \cos \theta_3 \\
& - (\sin^2 \theta_1 + \sin^2 \theta_2 + \sin^2 \theta_3)] \\
= & \frac{1}{\alpha^2} [m_2m_3 \sin^2 \theta_1 + m_3m_1 \sin^2 \theta_2 + m_1m_2 \sin^2 \theta_3], \tag{3.15}
\end{aligned}$$

where in the last equality, we used

$$\begin{aligned}
& 2 \cos \theta_1 \sin \theta_2 \sin \theta_3 + 2 \sin \theta_1 \cos \theta_2 \sin \theta_3 + 2 \sin \theta_1 \sin \theta_2 \cos \theta_3 \\
& = \sin \theta_1 (\cos \theta_2 \sin \theta_3 + \sin \theta_2 \cos \theta_3) + \sin \theta_2 (\cos \theta_1 \sin \theta_3 + \sin \theta_1 \cos \theta_3) \\
& \quad + \sin \theta_3 (\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2) \\
& = \sin \theta_1 \sin(\theta_2 + \theta_3) + \sin \theta_2 \sin(\theta_1 + \theta_3) + \sin \theta_3 \sin(\theta_1 + \theta_2) \\
& = \sin^2 \theta_1 + \sin^2 \theta_2 + \sin^2 \theta_3. \tag{3.16}
\end{aligned}$$

Thus we define

$$\alpha = \sqrt{m_2m_3 \sin^2 \theta_1 + m_3m_1 \sin^2 \theta_2 + m_1m_2 \sin^2 \theta_3}. \tag{3.17}$$

Now as in p.263 of [16], Section 11.2 of [11], we define

$$P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \quad Q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \quad Y = \begin{pmatrix} G \\ Z \\ W \end{pmatrix}, \quad X = \begin{pmatrix} g \\ z \\ w \end{pmatrix}, \tag{3.18}$$

where $p_i, q_i, i = 1, 2, 3$ and G, Z, W, g, z, w are all columns in \mathbf{R}^2 . We make the symplectic coordinate change

$$P = A^{-T}Y, \quad Q = AX, \tag{3.19}$$

where the matrix A is constructed as in the proof of Proposition 2.1 in [16]. Concretely, the matrix $A \in \mathbf{GL}(\mathbf{R}^6)$ is given by

$$A = \begin{pmatrix} I & A_1 & B_1 \\ I & A_2 & B_2 \\ I & A_3 & B_3 \end{pmatrix}, \tag{3.20}$$

as each A_i is a 2×2 matrix given by

$$A_1 = (a_1, Ja_1) = \begin{pmatrix} \frac{m_2 \cos \theta_2 \sin \theta_3 - m_3 \sin \theta_2 \cos \theta_3}{\alpha} & -\frac{(m_2 + m_3) \sin \theta_2 \sin \theta_3}{\alpha} \\ \frac{(m_2 + m_3) \sin \theta_2 \sin \theta_3}{\alpha} & \frac{m_2 \cos \theta_2 \sin \theta_3 - m_3 \sin \theta_2 \cos \theta_3}{\alpha} \end{pmatrix}, \tag{3.21}$$

$$A_2 = (a_2, Ja_2) = \begin{pmatrix} -\frac{m_1 \cos \theta_2 \sin \theta_3 + m_3 \sin(\theta_2 + \theta_3)}{\alpha} & \frac{m_1 \sin \theta_2 \sin \theta_3}{\alpha} \\ -\frac{m_1 \sin \theta_2 \sin \theta_3}{\alpha} & -\frac{m_1 \cos \theta_2 \sin \theta_3 + m_3 \sin(\theta_2 + \theta_3)}{\alpha} \end{pmatrix}, \tag{3.22}$$

$$A_3 = (a_3, Ja_3) = \begin{pmatrix} \frac{m_2 \sin(\theta_2 + \theta_3) + m_1 \sin \theta_2 \cos \theta_3}{\alpha} & \frac{m_1 \sin \theta_2 \sin \theta_3}{\alpha} \\ -\frac{m_1 \sin \theta_2 \sin \theta_3}{\alpha} & \frac{m_2 \sin(\theta_2 + \theta_3) + m_1 \sin \theta_2 \cos \theta_3}{\alpha} \end{pmatrix}. \tag{3.23}$$

To fulfill $A^T M A = I$ (cf. (13) in p.263 of [16]), we must have

$$B_1 = \rho_1(A_3 - A_2)^T = \frac{\rho_1 \sin \theta_1}{\alpha} I, \quad (3.24)$$

$$B_2 = \rho_2(A_1 - A_3)^T = -\frac{\rho_2 \sin \theta_2}{\alpha} R(\theta_3), \quad (3.25)$$

$$B_3 = \rho_3(A_2 - A_1)^T = -\frac{\rho_3 \sin \theta_3}{\alpha} R(-\theta_2), \quad (3.26)$$

where

$$\rho_i = \frac{\sqrt{m_1 m_2 m_3}}{m_i}, \quad \forall 1 \leq i \leq 3. \quad (3.27)$$

Moreover, from (3.24)-(3.26), we have

$$B_i B_j = B_j B_i \quad \forall 1 \leq i, j \leq 3. \quad (3.28)$$

Under the coordinate change (3.19), we get the kinetic energy

$$K = \frac{1}{2}(|G|^2 + |Z|^2 + |W|^2), \quad (3.29)$$

and the potential function

$$U(z, w) = \sum_{1 \leq i < j \leq 3} \frac{m_i m_j - e_i e_j}{d_{ij}(z, w)} = \sum_{1 \leq i < j \leq 3} \frac{m_i m_j \delta_{ij}}{d_{ij}(z, w)}, \quad (3.30)$$

with

$$d_{ij}(z, w) = |(A_i - A_j)z + (B_i - B_j)w| \quad \forall 1 \leq i < j \leq 3. \quad (3.31)$$

Let θ be the true anomaly. Then under the same steps of symplectic transformation in the proof of Lemma 3.1 of [16] (also in Theorem 11.10 of [11]), the resulting Hamiltonian function of the charged 3-body problem is given by

$$H(\theta, \bar{Z}, \bar{W}, \bar{z}, \bar{w}) = \frac{1}{2}(|\bar{Z}|^2 + |\bar{W}|^2) + (\bar{z} \cdot J \bar{Z} + \bar{w} \cdot J \bar{W}) + \frac{p - r(\theta)}{2p}(|\bar{z}|^2 + |\bar{w}|^2) - \frac{r(\theta)}{(\mu p)^{1/4}}, \quad (3.32)$$

where

$$r(\theta) = \frac{p}{1 + e \cos \theta} \quad (3.33)$$

and μ is given by (2.6). Note that the appearance of the term $(\mu p)^{1/4}$ and $p > 0$ require $\mu > 0$. From (2.18), we let

$$k = \frac{\sqrt[3]{\delta_{12}}}{|a_1 - a_2|} = \frac{\sqrt[3]{\delta_{23}}}{|a_2 - a_3|} = \frac{\sqrt[3]{\delta_{31}}}{|a_3 - a_1|}, \quad (3.34)$$

then together with (3.6) and (3.10), we have

$$\delta_{ij} = \frac{k^3 \sin^3 \theta_l}{\alpha^3}, \quad |a_i - a_j| = \frac{\sin \theta_l}{\alpha}, \quad (3.35)$$

where $\{i, j, l\}$ is any arbitrary permutation of $\{1, 2, 3\}$. Thus from (2.6) and (3.35), we have

$$\mu = \sum_{1 \leq i < j \leq 3, l \neq i, j} m_i m_j \frac{k^3 \sin^2 \theta_l}{\alpha^2} = \frac{k^3}{\alpha^2} \sum_{1 \leq i < j \leq 3, l \neq i, j} m_i m_j \sin^2 \theta_l = k^3, \quad (3.36)$$

where we used (3.17) in the last equality.

Based on Lemma 3.1 in [16], we now derive the transformed version of the elliptic triangle solutions and the linearized Hamiltonian system at such solutions. Let $\sigma = (p\mu)^{1/4}$ and $M = \text{diag}(m_1 I, m_2 I, m_3 I)$ as in (2.1) with $n = 3$ and $k = 2$.

Proposition 3.2 *Using notations (3.18), the elliptic triangle solution $(P(t), Q(t))^T$ of the system (1.2) with*

$$Q(t) = (r(t)R(\theta(t))a_1, r(t)R(\theta(t))a_2, r(t)R(\theta(t))a_3)^T, \quad P(t) = M\dot{Q}(t) \quad (3.37)$$

in the variable of time t , is transformed to the new solution $(Y(\theta), X(\theta))^T$ in the variable of the true anomaly θ with $G \equiv g \equiv 0$ with respect to the Hamiltonian function H of (3.32) given by

$$Y(\theta) = \begin{pmatrix} \bar{Z}(\theta) \\ \sigma \\ \bar{W}(\theta) \end{pmatrix} = \begin{pmatrix} 0 \\ \sigma \\ 0 \\ 0 \end{pmatrix}, \quad X(\theta) = \begin{pmatrix} \bar{z}(\theta) \\ \bar{w}(\theta) \end{pmatrix} = \begin{pmatrix} \sigma \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.38)$$

Proof. By transformation (3.19), the Lagrangian solution $(P(t), Q(t))^T$ in (3.37) is transformed to the solution $(0, Z(t), 0, 0, z(t), 0)^T$ of the Hamiltonian system with Hamiltonian function

$$H = \frac{1}{2}(Z_1^2 + Z_2^2 + W_1^2 + W_2^2) - U(z, w), \quad (3.39)$$

where

$$z(t) = \begin{pmatrix} r(t) \cos \theta(t) \\ r(t) \sin \theta(t) \end{pmatrix} \quad \text{and} \quad Z(t) = R(\theta(t)) \begin{pmatrix} \dot{r}(t) \\ r(t)\dot{\theta}(t) \end{pmatrix}. \quad (3.40)$$

Then setting $G = g = 0$, by the first transformation in the proof of Lemma 3.1 in [16], the solution $(Z(t), 0, z(t), 0)^T$ with respect to (3.39) is transformed to the solution $(\tilde{Z}, 0, \tilde{z}, 0)^T$ with respect to

$$H = \frac{1}{2}(\tilde{Z}_1^2 + \tilde{Z}_2^2 + \tilde{W}_1^2 + \tilde{W}_2^2) + (\tilde{z}_2 \tilde{Z}_1 - \tilde{z}_1 \tilde{Z}_2 + \tilde{w}_2 \tilde{W}_1 - \tilde{w}_1 \tilde{W}_2) \dot{\theta} - U(\tilde{z}, \tilde{w}), \quad (3.41)$$

where

$$\tilde{z}(t) = R(\theta(t))^T z(t) = \begin{pmatrix} r(t) \\ 0 \end{pmatrix}, \quad \tilde{Z}(t) = R(\theta(t))^T Z(t) = \begin{pmatrix} \dot{r}(t) \\ r(t)\dot{\theta}(t) \end{pmatrix}. \quad (3.42)$$

By the second transformation in the proof of Lemma 3.1 in [16], the solution $(\tilde{Z}, 0, \tilde{z}, 0)^T$ with respect to (3.41) is transformed to the solution $(\hat{Z}, 0, \hat{z}, 0)^T$ with respect to

$$\begin{aligned} H &= \frac{1}{2r^2}(\hat{Z}_1^2 + \hat{Z}_2^2 + \hat{W}_1^2 + \hat{W}_2^2) + (\hat{z}_2 \hat{Z}_1 - \hat{z}_1 \hat{Z}_2 + \hat{w}_2 \hat{W}_1 - \hat{w}_1 \hat{W}_2) \dot{\theta} \\ &\quad + \frac{r\ddot{r}}{2}(\hat{z}_1^2 + \hat{z}_2^2 + \hat{w}_1^2 + \hat{w}_2^2) - \frac{1}{r}U(\hat{z}, \hat{w}), \end{aligned} \quad (3.43)$$

where

$$\hat{z}(t) = \frac{1}{r(t)}\tilde{z}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{Z}(t) = r(t)\tilde{Z}(t) - \dot{r}(t)\tilde{z}(t) = \begin{pmatrix} 0 \\ r^2(t)\dot{\theta}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \sigma^2 \end{pmatrix}. \quad (3.44)$$

By the third transformation in the proof of Lemma 3.1 in [16], the independent variable t is transformed to the true anomaly θ , thus the solution $(\hat{Z}(t), 0, \hat{z}(t), 0)^T$ with respect to (3.43) is transformed to the solution $(\hat{Z}(\theta), 0, \hat{z}(\theta), 0)^T$ with respect to

$$\begin{aligned} H &= \frac{1}{2\sigma^2}(\hat{Z}_1^2 + \hat{Z}_2^2 + \hat{W}_1^2 + \hat{W}_2^2) + (\hat{z}_2 \hat{Z}_1 - \hat{z}_1 \hat{Z}_2 + \hat{w}_2 \hat{W}_1 - \hat{w}_1 \hat{W}_2) \\ &\quad + \frac{\mu(p - r(\theta))}{2\sigma^2}(\hat{z}_1^2 + \hat{z}_2^2 + \hat{w}_1^2 + \hat{w}_2^2) - \frac{r(\theta)}{\sigma^2}U(\hat{z}, \hat{w}), \end{aligned} \quad (3.45)$$

where

$$\hat{z}(\theta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{Z}(\theta) = \begin{pmatrix} 0 \\ \sigma^2 \end{pmatrix}. \quad (3.46)$$

By the last transformation in the proof of Lemma 3.1 in [16], the solution $(\hat{Z}(\theta), 0, \hat{z}(\theta), 0)^T$ with respect to (3.45) is transformed to the solution $(\bar{Z}(\theta), 0, \bar{z}(\theta), 0)^T$ with respect to (3.32), where

$$\bar{z}(t) = \sigma \hat{z} = \begin{pmatrix} \sigma \\ 0 \end{pmatrix}, \quad \bar{Z}(t) = \sigma^{-1} \hat{Z} = \begin{pmatrix} 0 \\ \sigma \end{pmatrix}. \quad (3.47)$$

This proves the Proposition. ■

Remark 3.3. As pointed out in [11], in the line 9 of p.273 in [16], the last term $-\frac{r}{\lambda p}S(\hat{z}, \hat{w})$ (with $\lambda = \mu$ and $S = U$ in our notations here) in the summand of the Hamiltonian function H is not correct and should be corrected to $-\frac{r}{(\mu p)^{1/4}}U(\bar{z}, \bar{w})$ as in our (3.32). In the line 11 of p.273 in [16], the stationary solution $(0, 1, 0, 0, 1, 0, 0, 0)^T$ is not correct and should be corrected to $(0, \sigma, 0, 0, \sigma, 0, 0, 0)^T$ as in our (3.38). Note that in general it may not be possible to have $\sigma = 1$, and it is the value σ which makes the following theorem holds.

We now derived the linearized Hamiltonian system at the elliptic solutions of the charged problem.

Proposition 3.4 *Using notations in (3.18), elliptic solution $(P(t), Q(t))^T$ of the system (1.2) with*

$$Q(t) = (r(t)R(\theta(t))a_1, r(t)R(\theta(t))a_2, r(t)R(\theta(t))a_3)^T, \quad P(t) = M\dot{Q}(t) \quad (3.48)$$

in time t with the matrix $M = \text{diag}(m_1, m_1, m_2, m_2, m_3, m_3)$, is transformed to the new solution $(Y(\theta), X(\theta))^T$ in the variable true anomaly θ with $G = g = 0$ with respect to the original Hamiltonian function H of (3.32) is given by

$$Y(\theta) = \begin{pmatrix} \bar{Z}(\theta) \\ \bar{W}(\theta) \end{pmatrix} = \begin{pmatrix} 0 \\ \sigma \\ 0 \\ 0 \end{pmatrix}, \quad X(\theta) = \begin{pmatrix} \bar{z}(\theta) \\ \bar{w}(\theta) \end{pmatrix} = \begin{pmatrix} \sigma \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.49)$$

Moreover, the linearized Hamiltonian system at the elliptic solution $\bar{\xi}_0 \equiv (Y(\theta), X(\theta))^T = (0, \sigma, 0, 0, \sigma, 0, 0, 0)^T \in \mathbf{R}^8$ depending on the true anomaly θ with respect to the Hamiltonian function H of (3.32) is given by

$$\dot{\zeta}(\theta) = JB(\theta)\zeta(\theta), \quad (3.50)$$

with

$$B(\theta) = H''(\theta, \bar{Z}, \bar{W}, \bar{z}, \bar{w})|_{\bar{\xi}=\bar{\xi}_0} = \begin{pmatrix} I & O & -J & O \\ O & I & O & -J \\ J & O & H_{\bar{z}\bar{z}}(\theta, \xi_0) & O \\ O & J & O & H_{\bar{w}\bar{w}}(\theta, \xi_0) \end{pmatrix}, \quad (3.51)$$

and

$$H_{\bar{z}\bar{z}}(\theta, \xi_0) = \begin{pmatrix} -\frac{2-e \cos \theta}{1+e \cos \theta} & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.52)$$

$$H_{\bar{w}\bar{w}}(\theta, \xi_0) = \begin{pmatrix} 1 - \frac{3d_1}{1+e \cos \theta} & -\frac{3d_2}{1+e \cos \theta} \\ -\frac{3d_3}{1+e \cos \theta} & 1 - \frac{3d_4}{1+e \cos \theta} \end{pmatrix}, \quad (3.53)$$

where

$$d_1 = m_1 \cos^2(\theta_2 - \theta_3) + m_2 \cos^2 \theta_2 + m_3 \cos^2 \theta_3, \quad (3.54)$$

$$d_2 = d_3 = m_1 \cos(\theta_2 - \theta_3) \sin(\theta_2 - \theta_3) + m_2 \cos \theta_2 \sin \theta_2 - m_3 \cos \theta_3 \sin \theta_3, \quad (3.55)$$

$$d_4 = m_1 \sin^2(\theta_2 - \theta_3) + m_2 \sin^2 \theta_2 + m_3 \sin^2 \theta_3. \quad (3.56)$$

H'' is the Hessian Matrix of H with respect to its variable \bar{Z} , \bar{W} , \bar{z} and \bar{w} . The corresponding quadratic Hamiltonian function is given by

$$\begin{aligned} H_2(\theta, \bar{Z}, \bar{W}, \bar{z}, \bar{w}) &= \frac{1}{2}|\bar{Z}|^2 + \bar{Z} \cdot J\bar{z} + \frac{1}{2}H_{\bar{z}\bar{z}}(\theta, \xi_0)|\bar{z}|^2 \\ &\quad + \frac{1}{2}|\bar{W}|^2 + \bar{W} \cdot J\bar{w} + \frac{1}{2}H_{\bar{w}\bar{w}}(\theta, \xi_0)|\bar{w}|^2. \end{aligned} \quad (3.57)$$

Proof. The proof is similar to the proof of Proposition 11.11 and Proposition 11.13 of [11]. We just need to compute $H_{\bar{z}\bar{z}}(\theta, \xi_0)$, $H_{\bar{z}\bar{w}}(\theta, \xi_0)$ and $H_{\bar{w}\bar{w}}(\theta, \xi_0)$.

For simplicity, we omit all the upper bars on the variables of H in (3.32) in this proof. By (3.32), we have

$$H_z = JZ + \frac{p-r}{p}z - \frac{r}{\sigma}U_z(z, w), \quad (3.58)$$

$$H_w = JW + \frac{p-r}{p}w - \frac{r}{\sigma}U_w(z, w), \quad (3.59)$$

and

$$\begin{cases} H_{zz} = \frac{p-r}{p}I - \frac{r}{\sigma}U_{zz}(z, w), \\ H_{zw} = H_{wz} = -\frac{r}{\sigma}U_{zw}(z, w), \\ H_{ww} = \frac{p-r}{p}I - \frac{r}{\sigma}U_{ww}(z, w), \end{cases} \quad (3.60)$$

where we write H_z and H_{zw} etc to denote the derivative of H with respect to z , and the second derivative of H with respect to z and then w respectively. Note that all the items above are 2×2 matrices.

Letting $U(z, w) = \sum_{1 \leq i < j \leq 3} U_{ij}(z, w)$, for $1 \leq i < j \leq 3$, we have

$$\begin{aligned} \frac{\partial U_{ij}}{\partial z}(z, w) &= -\frac{m_i m_j \delta_{ij} (A_i - A_j)^T [(A_i - A_j)z + (B_i - B_j)w]}{|(A_i - A_j)z + (B_i - B_j)w|^3}, \\ \frac{\partial U_{ij}}{\partial w}(z, w) &= -\frac{m_i m_j \delta_{ij} (B_i - B_j)^T [(A_i - A_j)z + (B_i - B_j)w]}{|(A_i - A_j)z + (B_i - B_j)w|^3}, \\ \frac{\partial^2 U_{ij}}{\partial z^2}(z, w) &= -\frac{m_i m_j \delta_{ij} (A_i - A_j)^T (A_i - A_j)}{|(A_i - A_j)z + (B_i - B_j)w|^3} \\ &\quad + 3 \frac{m_i m_j \delta_{ij} (A_i - A_j)^T [(A_i - A_j)z + (B_i - B_j)w] [(A_i - A_j)z + (B_i - B_j)w]^T (A_i - A_j)}{|(A_i - A_j)z + (B_i - B_j)w|^5}, \\ \frac{\partial^2 U_{ij}}{\partial z \partial w}(z, w) &= -\frac{m_i m_j \delta_{ij} (A_i - A_j)^T (B_i - B_j)}{|(A_i - A_j)z + (B_i - B_j)w|^3} \\ &\quad + 3 \frac{m_i m_j \delta_{ij} (A_i - A_j)^T [(A_i - A_j)z + (B_i - B_j)w] [(A_i - A_j)z + (B_i - B_j)w]^T (B_i - B_j)}{|(A_i - A_j)z + (B_i - B_j)w|^5}, \\ \frac{\partial^2 U_{ij}}{\partial w^2}(z, w) &= -\frac{m_i m_j \delta_{ij} (B_i - B_j)^T (B_i - B_j)}{|(A_i - A_j)z + (B_i - B_j)w|^3} \\ &\quad + 3 \frac{m_i m_j \delta_{ij} (B_i - B_j)^T [(A_i - A_j)z + (B_i - B_j)w] [(A_i - A_j)z + (B_i - B_j)w]^T (B_i - B_j)}{|(A_i - A_j)z + (B_i - B_j)w|^5}. \end{aligned}$$

Let

$$K = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.61)$$

Now evaluating these functions at the solution $\bar{\xi}_0 = (0, \sigma, 0, 0, \sigma, 0, 0, 0)^T \in \mathbf{R}^8$ and summing them with the lower indices, together with (3.21)-(3.28) and (3.35)-(3.36), we obtain

$$\begin{aligned}
\frac{\partial^2 U}{\partial z^2} \Big|_{\bar{\xi}_0} &= \sum_{1 \leq i < j \leq 3, l \neq i, j} \left(-\frac{m_i m_j \delta_{ij} \sin^2 \theta_l / \alpha^2}{\sigma^3 \sin^3 \theta_l / \alpha^3} I_2 + 3 \frac{m_i m_j \delta_{ij} \sin^4 \theta_l \sigma^2 / \alpha^4}{\sigma^5 \sin^5 \theta_l / \alpha^5} K_1 \right) \\
&= \sum_{1 \leq i < j \leq 3, l \neq i, j} \left(-\frac{m_i m_j k^3 \sin^5 \theta_l / \alpha^5}{\sigma^3 \sin^3 \theta_l / \alpha^3} I_2 + 3 \frac{m_i m_j k^3 \sin^7 \theta_l \sigma^2 / \alpha^7}{\sigma^5 \sin^5 \theta_l / \alpha^5} K_1 \right) \\
&= \frac{k^3}{\sigma^3 \alpha^2} \left(\sum_{1 \leq i < j \leq 3, l \neq i, j} m_i m_j \sin^2 \theta_l \right) (-I_2 + 3K_1) \\
&= \frac{k^3}{\sigma^3} K = \frac{\mu}{\sigma^3} K, \tag{3.62}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 U}{\partial z \partial w} \Big|_{\bar{\xi}_0} &= \sum_{1 \leq i < j \leq 3, l \neq i, j} \left(-\frac{m_i m_j \delta_{ij}}{\sigma^3 \sin^3 \theta_l / \alpha^3} (A_i - A_j)^T (B_i - B_j) \right. \\
&\quad \left. + 3 \frac{m_i m_j \delta_{ij} \sin^2 \theta_l \sigma^2 / \alpha^2}{\sigma^5 \sin^5 \theta_l / \alpha^5} K_1 (A_i - A_j)^T (B_i - B_j) \right) \\
&= \frac{k^3}{\sigma^3} (-I_2 + 3K_1) \left(\sum_{1 \leq i < j \leq 3, l \neq i, j} m_i m_j (A_i - A_j)^T (B_i - B_j) \right) \\
&= \frac{\mu}{\sigma^3} (-I_2 + 3K_1) [m_1 m_2 \left(-\frac{1}{\rho_3} B_3 \right) (B_1 - B_2) \\
&\quad + m_2 m_3 \left(-\frac{1}{\rho_1} B_1 \right) (B_2 - B_3) + m_1 m_3 \left(\frac{1}{\rho_2} B_2 \right) (B_1 - B_3)] \\
&= \frac{\mu}{\sigma^3} (-I_2 + 3K_1) \sqrt{m_1 m_2 m_3} [(-B_3 B_1 + B_1 B_3) \\
&\quad + (B_3 B_2 - B_2 B_3) + (-B_1 B_2 + B_2 B_1)] \\
&= O, \tag{3.63}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 U}{\partial w^2} \Big|_{\bar{\xi}_0} &= \sum_{1 \leq i < j \leq 3, l \neq i, j} \left(-\frac{m_i m_j \delta_{ij}}{\sigma^3 \sin^3 \theta_l / \alpha^3} (B_i - B_j)^T (B_i - B_j) \right. \\
&\quad \left. + 3 \frac{m_i m_j \delta_{ij} \sigma^2}{\sigma^5 \sin^5 \theta_l / \alpha^5} (B_i - B_j)^T (A_i - A_j) K_1 (A_i - A_j)^T (B_i - B_j) \right) \\
&= -\frac{k^3}{\sigma^3} \sum_{1 \leq i < j \leq 3, l \neq i, j} m_i m_j (B_i - B_j)^T (B_i - B_j) \\
&\quad + \frac{3k^3 \alpha^2}{\sigma^3} \sum_{1 \leq i < j \leq 3, l \neq i, j} \frac{m_i m_j}{\sin^2 \theta_l} (B_i - B_j)^T (A_i - A_j) K_1 (A_i - A_j)^T (B_i - B_j). \tag{3.64}
\end{aligned}$$

We firstly compute the first term of the right hand side of (3.64). Plugging (3.21) into $A^T M A = I$, we have

$$m_1 A_1^T A_1 + m_2 A_2^T A_2 + m_3 A_3^T A_3 = I_2. \tag{3.65}$$

Moreover, from (3.21)-(3.27), we have

$$B_1 - B_2 = \frac{1}{\rho_3} A_3^T, \tag{3.66}$$

$$B_2 - B_3 = \frac{1}{\rho_1} A_1^T, \quad (3.67)$$

$$B_3 - B_1 = \frac{1}{\rho_2} A_2^T. \quad (3.68)$$

Using (3.65)-(3.68), we have

$$\begin{aligned} & \sum_{1 \leq i < j \leq 3, l \neq i, j} m_i m_j (B_i - B_j)^T (B_i - B_j) \\ &= m_1 m_2 \frac{1}{\rho_3^2} A_3^T A_3 + m_2 m_3 \frac{1}{\rho_1^2} A_1^T A_1 + m_1 m_3 \frac{1}{\rho_2^2} A_2^T A_2 \\ &= m_3 A_3^T A_3 + m_1 A_1^T A_1 + m_2 A_2^T A_2 \\ &= I_2. \end{aligned} \quad (3.69)$$

We now compute the second term of the right hand side of (3.64). Let

$$\begin{aligned} D &= \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \\ &= \alpha^2 \sum_{1 \leq i < j \leq 3, l \neq i, j} \frac{1}{\sin^2 \theta_l} m_i m_j (B_i - B_j)^T (A_i - A_j) K_1 (A_i - A_j)^T (B_i - B_j). \end{aligned} \quad (3.70)$$

Then from (3.24)-(3.27), we have

$$\begin{aligned} D &= \frac{1}{\alpha^2} \left[\frac{m_1 m_2}{\sin^2 \theta_3} (\rho_1 \sin \theta_1 I_2 \right. \\ &\quad \left. + \rho_2 \sin \theta_2 R(-\theta_3)) \sin \theta_3 R(\theta_2) K_1 \sin \theta_3 R(-\theta_2) (\rho_1 \sin \theta_1 I_2 + \rho_2 \sin \theta_2 R(\theta_3)) \right. \\ &\quad \left. + \frac{m_1 m_3}{\sin^2 \theta_2} (\rho_1 \sin \theta_1 I_2 \right. \\ &\quad \left. + \rho_3 \sin \theta_3 R(\theta_2)) \sin \theta_2 R(-\theta_3) K_1 \sin \theta_2 R(\theta_3) (\rho_1 \sin \theta_1 I_2 + \rho_3 \sin \theta_3 R(-\theta_2)) \right. \\ &\quad \left. + \frac{m_2 m_3}{\sin^2 \theta_1} (-\rho_2 \sin \theta_2 R(-\theta_3) \right. \\ &\quad \left. + \rho_3 \sin \theta_3 R(\theta_2)) \sin \theta_1 I_2 K_1 \sin \theta_1 I_2 (-\rho_2 \sin \theta_2 R(\theta_3) + \rho_3 \sin \theta_3 R(-\theta_2)) \right] \\ &= \frac{1}{\alpha^2} \left[m_1 m_2 (\rho_1 \sin \theta_1 R(\theta_2) \right. \\ &\quad \left. + \rho_2 \sin \theta_2 R(\theta_2 - \theta_3)) K_1 (\rho_1 \sin \theta_1 R(-\theta_2) + \rho_2 \sin \theta_2 R(-\theta_2 + \theta_3)) \right. \\ &\quad \left. + m_1 m_3 (\rho_1 \sin \theta_1 R(-\theta_3) \right. \\ &\quad \left. + \rho_3 \sin \theta_3 R(\theta_2 - \theta_3)) K_1 (\rho_1 \sin \theta_1 R(\theta_3) + \rho_3 \sin \theta_3 R(-\theta_2 + \theta_3)) \right. \\ &\quad \left. + m_2 m_3 (-\rho_2 \sin \theta_2 R(-\theta_3) \right. \\ &\quad \left. + \rho_3 \sin \theta_3 R(\theta_2)) K_1 (-\rho_2 \sin \theta_2 R(\theta_3) + \rho_3 \sin \theta_3 R(-\theta_2)) \right]. \end{aligned} \quad (3.71)$$

Note that, for any $\varphi, \varphi_1, \varphi_2 \in \mathbf{R}$, we have

$$R(\varphi) K_1 R(-\varphi) = \begin{pmatrix} \cos^2 \varphi & \cos \varphi \sin \varphi \\ \cos \varphi \sin \varphi & \sin^2 \varphi \end{pmatrix}, \quad (3.72)$$

$$R(\varphi_1) K_1 R(-\varphi_2) + R(\varphi_2) K_1 R(-\varphi_1) = \begin{pmatrix} 2 \cos \varphi_1 \cos \varphi_2 & \sin(\varphi_1 + \varphi_2) \\ \sin(\varphi_1 + \varphi_2) & 2 \sin \varphi_1 \sin \varphi_2 \end{pmatrix}. \quad (3.73)$$

Using (3.71)-(3.73), we obtain

$$D_{11} = \frac{1}{\alpha^2} [m_1 m_2 (\rho_1^2 \sin^2 \theta_1 \cos^2 \theta_2 + \rho_2^2 \sin^2 \theta_2 \cos^2(\theta_2 - \theta_3))$$

$$\begin{aligned}
& +2\rho_1\rho_2\sin\theta_1\sin\theta_2\cos\theta_2\cos(\theta_2-\theta_3)) \\
& +m_1m_3(\rho_1^2\sin^2\theta_1\cos^2\theta_3+\rho_3^2\sin^2\theta_3\cos^2(\theta_2-\theta_3) \\
& +2\rho_1\rho_3\sin\theta_1\sin\theta_3\cos\theta_3\cos(\theta_2-\theta_3)) \\
& +m_2m_3(\rho_2^2\sin^2\theta_2\cos^2\theta_3+\rho_3^2\sin^2\theta_3\cos^2\theta_2 \\
& -2\rho_2\rho_3\sin\theta_2\sin\theta_3\cos\theta_2\cos\theta_3)] \\
= & \frac{1}{\alpha^2}[m_2^2m_3\sin^2\theta_1\cos^2\theta_2+m_1^2m_3\cos^2(\theta_2-\theta_3)\sin^2\theta_2 \\
& +m_2m_3^2\sin^2\theta_1\cos^2\theta_3+m_1^2m_2\cos^2(\theta_2-\theta_3)\sin^2\theta_3 \\
& +m_1m_3^2\sin^2\theta_2\cos^2\theta_3+m_1m_2^2\cos^2\theta_2\sin^2\theta_3] \\
& +\frac{m_1m_2m_3}{\alpha^2}[2\sin\theta_1\sin\theta_2\cos\theta_2\cos(\theta_2-\theta_3)+2\sin\theta_1\sin\theta_3\cos\theta_3\cos(\theta_2-\theta_3) \\
& -2\sin\theta_2\sin\theta_3\cos\theta_2\cos\theta_3] \\
= & \frac{1}{\alpha^2}[m_1\cos^2(\theta_2-\theta_3)(m_1m_3\sin^2\theta_2+m_1m_2\sin^2\theta_3) \\
& +m_2\cos^2\theta_2(m_1m_2\sin^2\theta_3+m_2m_3\sin^2\theta_1)+m_3\cos^2\theta_3(m_1m_3\sin^2\theta_2+m_2m_3\sin^2\theta_1)] \\
& +\frac{m_1m_2m_3}{\alpha^2}[\sin\theta_1\cos(\theta_2-\theta_3)(\sin\theta_2\cos\theta_2+\sin\theta_3\cos\theta_3) \\
& +\sin\theta_2\cos\theta_2(\sin\theta_1\cos(\theta_2-\theta_3)-\sin\theta_3\cos\theta_3) \\
& +\sin\theta_3\cos\theta_3(\sin\theta_1\cos(\theta_2-\theta_3)-\sin\theta_2\cos\theta_2)] \\
= & \frac{1}{\alpha^2}[m_1\cos^2(\theta_2-\theta_3)(\alpha^2-m_2m_3\sin^2\theta_1)+m_2\cos^2\theta_2(\alpha^2-m_1m_3\sin^2\theta_2) \\
& +m_3\cos^2\theta_3(\alpha^2-m_1m_2\sin^2\theta_3)] \\
& +\frac{m_1m_2m_3}{\alpha^2}[\sin\theta_1\cos(\theta_2-\theta_3)\frac{\sin 2\theta_2+\sin 2\theta_3}{2} \\
& +\sin\theta_2\cos\theta_2(\frac{\sin 2\theta_2+\sin 2\theta_3}{2}-\frac{\sin 2\theta_3}{2})+\sin\theta_3\cos\theta_3(\frac{\sin 2\theta_2+\sin 2\theta_3}{2}-\frac{\sin 2\theta_2}{2})] \\
= & m_1\cos^2(\theta_2-\theta_3)+m_2\cos^2\theta_2+m_3\cos^2\theta_3 \\
& -\frac{m_1m_2m_3}{\alpha^2}[\sin^2\theta_1\cos^2(\theta_2-\theta_3)+\sin^2\theta_2\cos^2\theta_2+\sin^2\theta_3\cos^2\theta_3] \\
& +\frac{m_1m_2m_3}{\alpha^2}[\sin\theta_1\sin(\theta_2+\theta_3)\cos^2(\theta_2-\theta_3)+\sin^2\theta_2\cos^2\theta_2+\sin^2\theta_3\cos^2\theta_3] \\
= & m_1\cos^2(\theta_2-\theta_3)+m_2\cos^2\theta_2+m_3\cos^2\theta_3. \tag{3.74}
\end{aligned}$$

Similarly, we have

$$D_{12}=D_{21} = d_2 = m_1\sin(\theta_2-\theta_3)\cos(\theta_2-\theta_3)+m_2\sin\theta_2\cos\theta_2-m_3\sin\theta_3\cos\theta_3, \tag{3.75}$$

$$D_{22} = d_4 = m_1\sin^2(\theta_2-\theta_3)+m_2\sin^2\theta_2+m_3\sin^2\theta_3. \tag{3.76}$$

By (3.62), (3.63), (3.64), (3.70), (3.74)-(3.76) and (3.60), we have

$$H_{zz}|_{\bar{\xi}_0} = \frac{p-r}{p}I - \frac{r\mu}{\sigma^4}K = I - \frac{r}{p}I - \frac{r\mu}{p\mu}K = I - \frac{r}{p}(I+K) = \begin{pmatrix} -\frac{2-e\cos\theta}{1+e\cos\theta} & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.77}$$

$$H_{zw}|_{\bar{\xi}_0} = -\frac{r}{\sigma}\frac{\partial^2 U}{\partial z\partial w}|_{\bar{\xi}_0} = O, \tag{3.78}$$

$$\begin{aligned}
H_{ww}|_{\bar{\xi}_0} &= \frac{p-r}{p}I - \frac{r\mu}{\sigma^4}\left(-I_2+3\begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}\right) = I - \frac{r}{p}I + \frac{r}{p}I - \frac{3r}{p}\begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \\
&= \begin{pmatrix} 1-\frac{3d_1}{1+e\cos\theta} & -\frac{3d_2}{1+e\cos\theta} \\ -\frac{3d_3}{1+e\cos\theta} & 1-\frac{3d_4}{1+e\cos\theta} \end{pmatrix}. \tag{3.79}
\end{aligned}$$

Thus the prof is complete. ■

Note that the linear Hamiltonian system (3.50) with the Hamiltonian function H_2 in (3.57) separates into two independent subsystem. The first one is in the variables $(\bar{Z}, \bar{z})^T \in \mathbf{R}^4$ with Hamiltonian function consists of the first line of H_2 in (3.57), which corresponds to the linearized system of the Kepler 2-body problem at Kepler orbits. The second one is in the variables $(\bar{W}, \bar{w})^T \in \mathbf{R}^4$ with Hamiltonian function consists of the second line of H_2 in (3.57) which depends on the central configuration strongly. The second part can be rewritten as follows in the variables $(\bar{W}, \bar{w})^T \in \mathbf{R}^4$:

$$\begin{pmatrix} \dot{\bar{W}} \\ \dot{\bar{w}} \end{pmatrix} = JB_2(\theta) \begin{pmatrix} \bar{W} \\ \bar{w} \end{pmatrix}, \quad (3.80)$$

with

$$\begin{aligned} B_2(\theta) &= \begin{pmatrix} I & -J \\ J & H_{\bar{w}\bar{w}}(\theta, \bar{\xi}_0) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 - \frac{3d_1}{1+e \cos \theta} & -\frac{3d_2}{1+e \cos \theta} \\ 1 & 0 & -\frac{3d_3}{1+e \cos \theta} & 1 - \frac{3d_4}{1+e \cos \theta} \end{pmatrix}. \end{aligned} \quad (3.81)$$

The reduced Hamiltonian system is given by the following result.

Theorem 3.5 *There exists a linear symplectic coordinate transform f generated by an orthogonal rotation matrix T depending only on the masses $m = (m_1, m_2, m_3)$ and the quantities of the charges e_1, e_2 , and e_3 such that under this transformation the lower right corner 2×2 sub-matrix in $B_2(\theta)$ of system (3.80) is diagonalized, and the coefficient matrix $B_2(\theta)$ of (3.80) becomes the matrix $\bar{B}_2(\theta)$ of (3.4).*

Proof. We rewrite the matrix $B_2(\theta)$ as following firstly.

$$B_2(\theta) = \begin{pmatrix} I & -J \\ J & \frac{e \cos \theta I + \tilde{D}}{1+e \cos \theta} \end{pmatrix}, \quad (3.82)$$

with

$$\tilde{D} = \begin{pmatrix} 1 - 3d_1 & -3d_2 \\ -3d_3 & 1 - 3d_4 \end{pmatrix}. \quad (3.83)$$

Then we have

$$\begin{aligned} \det \tilde{D} &= 1 - 3(d_1 + d_4) + 9(d_1 d_4 - d_2 d_3) \\ &= 1 - 3(m_1 \cos^2(\theta_2 - \theta_3) + m_2 \cos^2 \theta_2 + m_3 \cos^2 \theta_3 \\ &\quad + m_1 \sin^2(\theta_2 - \theta_3) + m_2 \sin^2 \theta_2 + m_3 \sin^2 \theta_3) \\ &\quad + 9[(m_1 \cos^2(\theta_2 - \theta_3) + m_2 \cos^2 \theta_2 + m_3 \cos^2 \theta_3)(m_1 \sin^2(\theta_2 - \theta_3) + m_2 \sin^2 \theta_2 + m_3 \sin^2 \theta_3) \\ &\quad - (m_1 \sin(\theta_2 - \theta_3) \cos(\theta_2 - \theta_3) + m_2 \sin \theta_2 \cos \theta_2 - m_3 \sin \theta_3 \cos \theta_3)^2] \\ &= 1 - 3(m_1 + m_2 + m_3) \\ &\quad + 9[m_1 m_2 (\cos^2(\theta_2 - \theta_3) \sin^2 \theta_2 + \sin^2(\theta_2 - \theta_3) \cos^2 \theta_2) \\ &\quad - 2 \sin(\theta_2 - \theta_3) \cos(\theta_2 - \theta_3) \sin \theta_2 \cos \theta_2) \\ &\quad + m_1 m_3 (\cos^2(\theta_2 - \theta_3) \sin^2 \theta_3 + \sin^2(\theta_2 - \theta_3) \cos^2 \theta_3) + 2 \sin(\theta_2 - \theta_3) \cos(\theta_2 - \theta_3) \sin \theta_3 \cos \theta_3) \\ &\quad + m_2 m_3 (\cos^2 \theta_2 \sin^2 \theta_3 + \sin^2 \theta_2 \cos^2 \theta_3 + 2 \sin \theta_2 \cos \theta_2 \sin \theta_3 \cos \theta_3)] \end{aligned}$$

$$\begin{aligned}
&= -2 + 9[m_1 m_2 (\cos(\theta_2 - \theta_3) \sin \theta_2 - \sin(\theta_2 - \theta_3) \cos \theta_2)^2 \\
&\quad + m_1 m_3 (\cos(\theta_2 - \theta_3) \sin \theta_3 + \sin(\theta_2 - \theta_3) \cos \theta_3)^2 + m_2 m_3 (\cos \theta_2 \sin \theta_3 + \sin \theta_2 \cos \theta_3)^2] \\
&\quad - 2 + 9[m_1 m_2 \sin^2 \theta_3 + m_1 m_3 \sin^2 \theta_2 + m_2 m_3 \sin^2(\theta_2 + \theta_3)] \\
&\quad - 2 + 9\alpha^2 - 2 + \frac{\beta}{4}.
\end{aligned} \tag{3.84}$$

Then the characteristic polynomial of \tilde{D} is

$$\det(\tilde{D} - \lambda I) = \lambda^2 - [2 - 3(d_1 + d_4)]\lambda + \det \tilde{D} = \lambda^2 + \lambda - 2 + \frac{\beta}{4}. \tag{3.85}$$

Thus the two eigenvalues of \tilde{D} are

$$\lambda_1 = \frac{-1 - \sqrt{9 - \beta}}{2}, \quad \lambda_2 = \frac{-1 + \sqrt{9 - \beta}}{2}. \tag{3.86}$$

Next as in the proof of Theorem 11.14 of [11], we denote the orthonormal eigenvectors of \tilde{D} belonging to the eigenvalues λ_1 and λ_2 by ξ_1 and ξ_2 respectively. Let $A = (\xi_1, \xi_2)$ be the 2×2 matrix formed by ξ_1 and ξ_2 as its column vectors. Then we obtain

$$A^T \tilde{D} A = \begin{pmatrix} \lambda_1 |\xi_1|^2 & 0 \\ 0 & \lambda_2 |\xi_2|^2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{3.87}$$

Replacing \tilde{D} by I in (3.87) yields the fact $A \in \text{SO}(2)$, and then $A^{-1} = A^T$ and $\hat{A} = \text{diag}(A, A) \in \text{Sp}(4)$ hold.

Now we define the coordinate transformation by

$$\bar{W} = AW, \quad \bar{w} = Aw.$$

Thus the system (3.80) becomes

$$\begin{aligned}
\begin{pmatrix} \dot{W}(\theta) \\ \dot{w}(\theta) \end{pmatrix} &= \hat{A}^{-1} J \bar{B}_2(\theta) \hat{A} \begin{pmatrix} W(\theta) \\ w(\theta) \end{pmatrix} \\
&= J \hat{A}^T \bar{B}_2(\theta) \hat{A} \begin{pmatrix} W(\theta) \\ w(\theta) \end{pmatrix} \\
&= J \begin{pmatrix} A^T & 0 \\ 0 & A^T \end{pmatrix} \begin{pmatrix} I & -J \\ J & \frac{1}{1+e \cos \theta} [(e \cos \theta) I + \tilde{D}] \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} W(\theta) \\ w(\theta) \end{pmatrix} \\
&= J \begin{pmatrix} I & -J \\ J & \frac{1}{1+e \cos \theta} [(e \cos \theta) I + A^T \tilde{D} A] \end{pmatrix} \begin{pmatrix} W(\theta) \\ w(\theta) \end{pmatrix}.
\end{aligned}$$

Together with (3.87), we obtain $B_2(\theta)$ in (3.80) as claimed. Then we obtain $\bar{B}_2(\theta)$ in (3.4) as claimed. ■

The proof of Theorem 1.1 is complete.

4 Appendix. The range of β

Lemma A. For $m = (m_1, m_2, m_3) \in (\mathbf{R}^+)^3$ and $\theta_1, \theta_2, \theta_3$ being the three inner angles of some triangle, the number β defined by (1.4) has range $[0, 9]$.

Proof. Note that $\beta = 9$ when $m_i = 1/3$ and $\theta_i = \pi/3$ for $1 \leq i \leq 3$. Thus it suffices to prove $\beta \leq 9$. Without loss of generality, we suppose (2.22) holds, i.e., $m_1 + m_2 + m_3 = 1$.

Let a, b, c be the three edges opposite to $\theta_1, \theta_2, \theta_3$ respectively and R be the radius of the circumscribed circle of the triangle. Moreover, let

$$\lambda_1 = \sin^2 \theta_1, \quad \lambda_2 = \sin^2 \theta_2, \quad \lambda_3 = \sin^2 \theta_3, \quad (4.1)$$

and

$$f(m) = \lambda_1 m_2 m_3 + \lambda_2 m_3 m_1 + \lambda_3 m_1 m_2. \quad (4.2)$$

Then for all admissible m and λ_i s we have

$$0 \leq f(m) \leq 3. \quad (4.3)$$

Thus we need to find maximal value of the function $f(m)$ in the area $m = (m_1, m_2, m_3) \in \mathbf{R}^3$ with the constraint (2.22) for $m_i \in [0, 1]$ with $1 \leq i \leq 3$ and parameters λ_i for $1 \leq i \leq 3$.

If the maximal value of f is obtained at the boundary of the constraint area, then $m_i = 0$ holds for at least one i . Without loose of generality, we suppose $m_1 = 0$. Then for such m we get

$$f(m) = \lambda_1 m_2 m_3 \leq \frac{1}{4} \lambda_1 (m_2 + m_3)^2 = \frac{1}{4} \lambda_1 = \frac{1}{4} \sin^2 \theta_1 \leq \frac{1}{4}, \quad (4.4)$$

and $\beta \leq 36f(m) \leq 9$ holds.

If the maximal value of f is obtained in the interior of the area, we introduce the Lagrangian multiplier λ , and let

$$F(m, \lambda) = \lambda_1 m_2 m_3 + \lambda_2 m_3 m_1 + \lambda_3 m_1 m_2 - \lambda(m_1 + m_2 + m_3 - 1).$$

Then in addition to (2.22) we have

$$\begin{cases} \lambda_3 m_2 + \lambda_2 m_3 - \lambda = 0, \\ \lambda_3 m_1 + \lambda_1 m_3 - \lambda = 0, \\ \lambda_2 m_1 + \lambda_1 m_2 - \lambda = 0. \end{cases} \quad (4.5)$$

Solving it, we obtain

$$\begin{cases} m_1^* = \frac{\lambda_2 + \lambda_3 - \lambda_1}{2\lambda_2\lambda_3} \lambda, \\ m_2^* = \frac{\lambda_1 + \lambda_3 - \lambda_2}{2\lambda_3\lambda_1} \lambda, \\ m_3^* = \frac{\lambda_1 + \lambda_2 - \lambda_3}{2\lambda_1\lambda_2} \lambda, \end{cases} \quad (4.6)$$

where we use $m^* = (m_1^*, m_2^*, m_3^*)$ to denote the critical point produced by the system (4.5). Plugging (4.6) into (2.22), we obtain

$$\lambda = \frac{2\lambda_1\lambda_2\lambda_3}{S},$$

where

$$S = 2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2). \quad (4.7)$$

Note that we have $S > 0$. In fact, from

$$a = 2R \sin \theta_1, \quad b = 2R \sin \theta_2, \quad c = 2R \sin \theta_3,$$

we obtain

$$\begin{aligned} S &= 2\left(\frac{a^2b^2}{16R^4} + \frac{b^2c^2}{16R^4} + \frac{c^2a^2}{16R^4}\right) - \left(\frac{a^4}{16R^4} + \frac{b^4}{16R^4} + \frac{c^4}{16R^4}\right) \\ &= \frac{2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)}{16R^4} \\ &= \frac{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}{16R^4} \\ &> 0. \end{aligned}$$

Thus m^* is the unique critical point of f under the constraint (2.22).

Now for this solution m^* , $m_1^* > 0$ implies

$$\begin{aligned}
0 &< m_1^* \\
&= \frac{\lambda_1(\lambda_2 + \lambda_3 - \lambda_1)}{S} \\
&= \frac{\lambda_1}{S} \left(\frac{b^2}{4R^2} + \frac{c^2}{4R^2} - \frac{a^2}{4R^2} \right) \\
&= \frac{\lambda_1(b^2 + c^2 - a^2)}{4R^2 S} \\
&= \frac{2\lambda_1 bc \cos \theta_1}{4R^2 S}.
\end{aligned}$$

Thus θ_1 is an acute angle. By the same reason, that m_2^* and $m_3^* > 0$ implies that θ_2 and θ_3 are acute angles too. Therefore, the solution point m^* given by (4.6) is located in the interior of the constraint area if and only if the given triangle is an acute triangle. At such a point m^* , we then obtain

$$\begin{aligned}
f(m^*) &= \lambda_1 \frac{\lambda_2(\lambda_1 + \lambda_3 - \lambda_2)}{S} \frac{\lambda_3(\lambda_1 + \lambda_2 - \lambda_3)}{S} + \lambda_2 \frac{\lambda_3(\lambda_1 + \lambda_2 - \lambda_3)}{S} \frac{\lambda_1(\lambda_2 + \lambda_3 - \lambda_1)}{S} \\
&\quad + \lambda_3 \frac{\lambda_1(\lambda_2 + \lambda_3 - \lambda_1)}{S} \frac{\lambda_2(\lambda_1 + \lambda_3 - \lambda_2)}{S} \\
&= \frac{\lambda_1 \lambda_2 \lambda_3}{S^2} [\lambda_1^2 - \lambda_2^2 - \lambda_3^2 + 2\lambda_2 \lambda_3 + \lambda_2^2 - \lambda_3^2 - \lambda_1^2 + 2\lambda_1 \lambda_3 + \lambda_3^2 - \lambda_1^2 - \lambda_2^2 + 2\lambda_1 \lambda_2] \\
&= \frac{\lambda_1 \lambda_2 \lambda_3}{S},
\end{aligned} \tag{4.8}$$

where we have used (4.7) in the last equality. However, by (4.1) and (4.7), we have

$$\begin{aligned}
&\lambda_1 \lambda_2 \lambda_3 - \frac{1}{4} S \\
&= \lambda_1 \lambda_2 \lambda_3 - \frac{1}{4} [2(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1) - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)] \\
&= \lambda_1 \lambda_2 \lambda_3 - \lambda_1 \lambda_2 + \frac{1}{4} (\lambda_1 + \lambda_2 - \lambda_3)^2 \\
&= \sin^2 \theta_1 \sin^2 \theta_2 (\sin^2 \theta_3 - 1) + \frac{1}{4} (\sin^2 \theta_1 + \sin^2 \theta_2 - \sin^2(\theta_1 + \theta_2))^2 \\
&= -\sin^2 \theta_1 \sin^2 \theta_2 \cos^2 \theta_3 \\
&\quad + \frac{1}{4} (\sin^2 \theta_1 + \sin^2 \theta_2 - \sin^2 \theta_1 \cos^2 \theta_2 + \cos^2 \theta_1 \sin^2 \theta_2 - 2 \sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2)^2 \\
&= -\sin^2 \theta_1 \sin^2 \theta_2 \cos^2 \theta_3 + \frac{1}{4} [-2 \sin \theta_1 \sin \theta_2 \cos(\theta_1 + \theta_2)]^2 \\
&= 0.
\end{aligned} \tag{4.9}$$

Thus from (4.8) and (4.9), we have $f(m^*) = \frac{1}{4}$. Together with (4.4) and the uniqueness of m^* as critical point of f under the constraint (2.22), $\frac{1}{4}$ must be the maximal value of f . Hence we obtain $\beta \leq 36f(m^*) = 9$. ■

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